Basic Elasticity Equations

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1 Basic elasticity equations in general form

In the general three-dimensional elasticity problem there are 15 unknown quantities (6 cartesian stress components, 6 cartesian strain components and 3 displacement components) which must be determined at every point in the body. We have 15 independent equations [1] (3 stress equations of equilibrium, 6 strain-displacement relations and 6 stress-strain relations):

1.1 3 stress equations of equilibrium

$$
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z &= 0
\end{aligned}
$$

[1] In addition boundary conditions must also be satisfied.

1.2 6 strain-displacement relations

$$\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} \\
\epsilon_{zz} &= \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{aligned}$$

1.3 6 stress-strain displacement relations for isotropic material

$$\left\{\begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{array}\right\} = \left[\begin{array}{cccccc} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{array}\right] \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{array}\right\}$$

1.4 6 compatibility equations

Ensure that given field of strain is compatible, because for given field of displacement strain field can be unambigouesly calculated using strain-displacement equations but not vice versa. Recall assembling big piece of paper from incompatible small pieces as unresolvable task.

$$\begin{aligned}
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial z \partial x} \\
2\frac{\partial^2 \epsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
2\frac{\partial^2 \epsilon_y}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
2\frac{\partial^2 \epsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)
\end{aligned}$$

1.5 Plane stress and plane strain

Plane stress is such a stress state that stress components are only in 2 dimensions (very thin plate):

$$\sigma_{xx} \neq 0, \sigma_{yy} \neq 0, \sigma_{zz} = 0, \tau_{xy} \neq 0, \tau_{xz} = 0, \tau_{yz} = 0$$

Similarly, plane strain has strain component in 2 dimensions (slice cut from dam of infinite length):

$$\epsilon_{xx} \neq 0, \epsilon_{yy} \neq 0, \epsilon_{zz} = 0, \gamma_{xy} \neq 0, \gamma_{xz} = 0, \gamma_{yz} = 0$$

1.6 Stress function approach

Let's get acquainted with Airy's stress function first which enables us to solve 2D problem. Airy's stress function Φ is defined such that:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} \quad (1)$$

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \quad (2)$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \quad (3)$$

Number of 15 elasticity equations for general 3D case, which need to be solved, is reduced to 3. 2 equilibrium equations are satisfied identically by the stress function $\Phi(x, y)$ and after substituting into compatibility equation from Eq's (1),(2),(3) we obtain following relationship for Φ:

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = \nabla^4 \Phi = 0$$